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J.A. BERGSTRA & J.W. KLOP

INITIAL ALGEBRA SPECIFICATIONS FOR PARAMETRIZED DATA TYPES

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Initial algebra specifications for parametrized data types

by

J.A. Bergstra*) & J.W. Klop

ABSTRACT

We consider parametrized data types ϕ: $\mathrm{Alg}(\Sigma) \to \mathrm{Alg}(\Delta)$ where ϕ is a partial functor from the class of all Σ-algebras (the parameter algebras) to the class of Δ-algebras (the target algebras), for given signatures Σ,Δ with Δ extending Σ. Here it is required that the target algebra is generated by a homomorphic image of the parameter algebra.

For such parametrized data types we prove a general theorem about the existence of initial algebra specifications with conditional equations. The theorem involves the concept of an effectively given parametrized data type.

KEY WORDS & PHRASES: *initial algebra specifications, parametrized data-type, semi-computable data type*

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*) Department of Computer Science, University of Leiden, Wassenaarseweg 80, 2300 RA LEIDEN, The Netherlands

0. INTRODUCTION

We will discuss the specification theory for persistent parametrized data types according to the definitions in ADJ [9].

Our aim is to propose a general necessary and sufficient condition for the existence of an algebraic specification for a given parametrized data type.

We call a persistent parametrized data type ϕ: Alg(Σ) $\to$ Alg(Δ) *effective* if there exists a uniform algorithm which transforms finite specifications for parameter algebras into finite specifications for target algebras. Especially interesting is the case that Dom(ϕ) contains all and only semi-computable algebras in a quasi-variety Alg(Σ,E) with E finite.

For such ϕ we show that ϕ is effective if and only if ϕ possesses an algebraic specification (Δ,F) with F an r.e. set of conditional equations.

The following comments are in order.

(i) Of course the definitions of a parametrized data type and its specification as employed here, are by no means the only ones. For further information we refer to the following papers: [5,6,7,8,10]

(ii) We preferred not to use the full formalism of category theory; instead we introduce a parametrized data type ϕ: Alg(Σ) $\to$ Alg(Δ) as a ternary relation containing triples $(\mathcal{A},\alpha,\mathcal{B})$ where $\mathcal{A} \in$ Alg(Σ), $\mathcal{B} \in$ Alg(Δ) and α: $\mathcal{A} \to \mathcal{B}|_{\Sigma}$ is a homomorphism such that
(1) the relation is closed under taking isomorphic copies of parameter and target algebras, and
(2) if $(\mathcal{A},\alpha_1,\mathcal{B}_1)$ and $(\mathcal{A},\alpha_2,\mathcal{B}_2) \in \phi$ then $\mathcal{B}_1 \cong \mathcal{B}_2$.

(iii) If one allows auxiliary sorts and functions it is possible to prove that a specification (Δ,F) with F an r.e. set can be transformed into an equivalent but finite specification (Γ,H) with $\Gamma \supseteq \Delta$ and H finite. A similar result is obtained in BERGSTRA-KLOP [1] .

(iv) This paper uses a result derived in BERGSTRA-KLOP [1] about the specification of effective parametrized data types with a domain consisting of *minimal* input algebras only.

1. PRELIMINARIES

1.1. Signatures and algebras.

A *signature* is a triple consisting of three listings, one of *sorts*, one of *functions* and one of *constants*. If Σ, Δ are signatures, the meaning of $\Sigma \subseteq \Delta$, $\Sigma \cup \Delta$ and $\Sigma \cap \Delta$ is clear.

The notation of a Σ-*algebra* is well-known, and will not be in extenso be repeated here. We conceive a Σ-algebra as a triple containing Σ, an algebra A and an interpretation function telling us what domains A_s in A correspond to the sorts s in Σ, and what functions and constants in A correspond to the function - and constant symbols in Σ. The set of Σ-*terms* is $\mathrm{Ter}(\Sigma)$; the set of *closed* Σ-terms is $\mathrm{Ter}^c(\Sigma)$. (A term is closed if it contains no variables.) The class of all Σ-algebras is $\mathrm{Alg}(\Sigma)$, and the class of all *minimal* Σ-algebras is $\mathrm{ALG}(\Sigma)$. Here an algebra A is a minimal if it contains no proper subalgebras, equivalently, if A is isomorphic ($\cong$) to a quotient of a term algebra, equivalently if every element a in $A \in \mathrm{ALG}(\Sigma)$ is the denotation of a Σ-term.

The concept of a *homomorphism* α between algebras A_1, A_2 of the same signature is standard. It goes without explicit mention that every map in this paper $\alpha: A_1 \to A_2$ where $A_1, A_2 \in \mathrm{Alg}(\Sigma)$, is a homomorphism.

If $\Sigma \subseteq \Sigma'$ and $A' \in \mathrm{Alg}(\Sigma')$, then $A = A'|_\Sigma$ is the *restriction* of A' to the signature Σ. In this case A' is also called an *expansion* of A. The following '*Joint Expansion Property*' is easily verified:

if $A_i \in \mathrm{Alg}(\Sigma_i)$, $i = 0,1,2$, such that $\Sigma_1 \cap \Sigma_2 = \Sigma_0$ and moreover $A_{1,s} \cap A_{2,s'} = \emptyset$ for all $s \in \Sigma_1 - \Sigma_0$, $s' \in \Sigma_2 - \Sigma_0$, then there is a unique expansion $A_1 \sqcup A_2 \in \mathrm{Alg}(\Sigma_1 \cup \Sigma_2)$ of A_1 and A_2.

$$\begin{tikzcd}[column sep=small] & A_1 \sqcup A_2 \arrow[dl, dash] \arrow[dr, dash] & \\ A_1 \arrow[dr, dash] & & A_2 \arrow[dl, dash] \\ & A_0 & \end{tikzcd}$$

Instead of $\gamma: A \to B|_\Sigma$ for $A \in \mathrm{Alg}(\Sigma)$, $B \in \mathrm{Alg}(\Delta)$, $\Sigma \subseteq \Delta$, we will often use the triple notation (A, γ, B). Triples (A_i, γ_i, B_i) $i = 1,2$, $A_i \in \mathrm{Alg}(\Sigma)$, $B_i \in \mathrm{Alg}(\Delta)$, $\Sigma \subseteq \Delta$, are called *congruent* if there are isomorphisms α, β

making the following diagram commute:

$$\begin{tikzcd} A_1 \arrow[r, "\gamma_1"] \arrow[d, "\alpha"'] & B_1 \arrow[d, "\beta"] \\ A_2 \arrow[r, "\gamma_2"'] & B_2 \end{tikzcd}$$

An important construction is the following one: Let $\Gamma \subseteq \Delta$ and $B \in \mathrm{Alg}(\Delta)$. Furthermore, let $A \subseteq \bigcup_{s \in \underline{sorts}(\Gamma)} B_s$, where B_s is the domain in B corresponding to sort s. Then $\langle B,\Gamma,A\rangle$ is the *subalgebra generated by* A *in* B *by means of* Γ (i.e. by the Γ-operations and Γ-constants).

In particular, if $A \in \mathrm{Alg}(\Sigma)$ with $\Sigma \subseteq \Gamma$ and $A = \bigcup_{s \in \underline{sorts}(\Sigma)} A_s$, then we write also $\langle B,\Gamma,A\rangle$ instead of $\langle B,\Gamma,A\rangle$.

1.2. Specifications of algebras.

In this paper we will be interested in subclasses of $\mathrm{Alg}(\Sigma)$ of the form $\mathrm{Alg}(\Sigma,E) = \{A \in \mathrm{Alg}(\Sigma) \mid A \models E\}$, where E is a set of *conditional equations*. A conditional equation has the form

$$s_1 = t_1 \wedge \ldots \wedge s_k = t_k \rightarrow s = t$$

for some $k \geq 0$ and $s,t,s_i,t_i (i = 1,\ldots,k) \in \mathrm{Ter}(\Sigma)$. The conditional equation is *closed* if all terms in it are closed.

The unique initial term algebra of signature Σ satisfying the set E of conditional equations, is denoted by $I(\mathrm{Alg}(\Sigma,E))$. It is a representant of the isomorphism class of initial algebras in $\mathrm{Alg}(\Sigma,E)$. Isomorphism is denoted by $\cong$.

If E is a set of conditional equations, E° denotes the set of closed equations (so without conditions) derivable from E. An example of such a set of closed Σ-equations is the *congruence* $\equiv_A$ corresponding to a minimal algebra $A \in \mathrm{ALG}(\Sigma)$; that is, the set of all closed Σ-equations true in A.

If $A \in \mathrm{Alg}(\Sigma)$ and for some (Σ',E') with $\Sigma' \supseteq \Sigma$ it is the case that $A \cong A'|_{\Sigma}$ where $A' = I(\mathrm{Alg}(\Sigma',E'))$, then we say that A can be *specified* (using auxiliary sorts and functions) by (Σ',E').

<u>Notation</u>: $(\Sigma',E') \xrightarrow{\Sigma} A$.

To give an actual specification of A by (Σ',E') we will insist that also the isomorphism α: $A \to A'|_{\Sigma}$ is mentioned.

<u>Notation</u>: $(\Sigma',E') \underset{\alpha}{\xrightarrow{\Sigma}} A$. So $(\Sigma',E') \xrightarrow{\Sigma} A$ is in fact short for $\exists\alpha\ (\Sigma',E) \underset{\alpha}{\xrightarrow{\Sigma}} A$.

1.3. <u>Semi-computable algebras</u>.

Notation: If $w = s_1 \times \ldots \times s_k$, where $s_i \in$ <u>sorts</u> (Σ), $i = 1,\ldots,k$, then X_w abbreviates $X_{s_1} \times \ldots \times X_{s_k}$.

The following definition is standard: $A \in \mathrm{Alg}(\Sigma)$ is *effectively presented* if corresponding to the domains $A_s (s \in \mathrm{sorts}(\Sigma))$ there are mutually disjoint recursive sets Ω_s and surjective maps $\alpha_s : \Omega_s \longrightarrow A_s (s \in \underline{\mathrm{sorts}}(\Sigma))$, such that for each function F in A of type $w \longrightarrow s$, there is a recursive f: $\Omega_w \longrightarrow \Omega_s$ which commutes the diagram:

$$\begin{array}{ccc} A_w & \xrightarrow{F} & A_s \\ \uparrow{\scriptstyle \alpha_w} & & \uparrow{\scriptstyle \alpha_s} \\ \Omega_w & \xrightarrow{f} & \Omega_s \end{array}$$

where $\alpha_w(x_1,\ldots,x_k) = (\alpha_{s_1}(x_1),\ldots,\alpha_{s_k}(x_k))$.

Now A is *semi-computable* ($A \in \mathrm{Sca}(\Sigma)$) if in addition for each $s \in$ <u>sorts</u> (Σ) the relation $\equiv_{\alpha_s}$ defined on Ω_s by

$$a \equiv_{\alpha_s} a' \iff \alpha_s(a) = \alpha_s(a')$$

is r.e..

We will need the following fact, proved in BERGSTRA-TUCKER [2]:

1.3.1. <u>LEMMA</u>. A *is semi-computable iff* A *has a finite specification.*

2. PARAMETRIZED DATA TYPES

For signatures Σ and Δ with $\Sigma \subseteq \Delta$, a parametrized data type ϕ: Alg(Σ) $\longrightarrow$ Alg(Δ) is a class of triples (A,γ,B) where $A \in \mathrm{Alg}(\Sigma)$, $B \in \mathrm{Alg}(\Delta)$ and $\gamma\colon A \longrightarrow B_\Sigma$ is a surjective homomorphism such that $B = \langle B,\Delta,B_\Sigma \rangle$ (i.e. $\phi(A)$ generates B).

Furthermore, the class ϕ must satisfy the following global conditions:

(i) if $(A,\gamma,B) \in \phi$ and $(A',\gamma',B') \in \phi$ is congruent with (A,γ,B), then $(A',\gamma',B') \in \phi$;

(ii) if $(A,\gamma,B) \in \phi$, $(A',\gamma',B') \in \phi$ and $\alpha\colon A \to A'$ is an (injective) homomorphism, then there is an (injective) homomorphism $\beta\colon B \to B'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\gamma"'] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "\gamma'"'] & B' \end{tikzcd}$$

commutes.

Furthermore, ϕ is called *persistent* if for all $(A,\gamma,B) \in \phi$ the homomorphism γ is injective as well as surjective.

2.1. Effectively given parametrized data types.

Let (σ,ε) be a monotonic partial recursive transformation of finite specifications, transforming (Σ',E') into $(\sigma(\Sigma',E'), \varepsilon(\Sigma',E')) = (\Sigma'',E'')$. Here the monotonicity requirement is that $\Sigma'' \supseteq \Sigma'$ and $E'' \supseteq E'$.

Now we say that a parametrized data type ϕ: Alg(Σ) $\to$ Alg(Δ) is *effectively given* by (σ,ε) if for each triple $(A,\gamma,B) \in \phi$ and for each finite specification $(\Sigma',E') \xrightarrow{\Sigma} A$ the following triple (A',γ',B') is congruent to (A,γ,B):

$$A' = I(\mathrm{Alg}(\Sigma',E'))|_\Sigma$$

$$B' = I(\mathrm{Alg}(\Sigma'',E''))|_\Delta$$

γ': $A' \to B'|_\Sigma$ is the homomorphism induced by the unique homomorphism
$\imath$: $I(Alg(\Sigma',E')) \longrightarrow I(Alg(\Sigma'',E''))|_{\Sigma'}$.

In a diagram:

$$\begin{tikzcd} I(Alg(\Sigma',E')) \arrow[r, "\imath"] \arrow[d, "\Sigma"] & I(Alg(\Sigma'',E''))|_{\Sigma'} \arrow[d, "\Sigma"] \\ A' = I(Alg(\Sigma'E')) \arrow[r, "\gamma'"'] & I(Alg(\Sigma'',E''))|_\Sigma = B'|_\Sigma \end{tikzcd}$$

2.2. Algebraically specified parametrized data types.

ϕ: $Alg(\Sigma) \longrightarrow Alg(\Delta)$ has an *algebraic specification* if there is a specification (Γ,H) such that for each $(A,\gamma,B) \in \phi$ and for each specification $(\Sigma',E') \xrightarrow{\Sigma} A$ (with $\Sigma' \cap (\Gamma \cup \Delta) = \Sigma$) the following triple (A',γ',B') is congruent to (A,γ,B):

$$A' = I(Alg(\Sigma',E'))$$

$$B' = I(Alg(\Sigma' \cup \Gamma, E' \cup E))$$

γ' again induced by the unique homomorphism

$$\imath : I(Alg(\Sigma',E')) \longrightarrow I(Alg(\Sigma'',E''))|_{\Sigma'}.$$

The following lemma will play a key role in the sequel.

2.3. LEMMA. *Suppose that* ϕ: $Alg(\Sigma) \longrightarrow Alg(\Delta)$ *is persistent and effectively given by* (σ,ε) *with* $Dom(\phi) = ALG(\Sigma,E) \cap Sca(\Sigma)$ *for some finite* E.

Then ϕ *has an algebraic specification* (Δ,H) *where* H *is a (possible infinite) set of closed conditional equations.*

Moreover H *is r.e., uniformly in recursive indices for* (σ,ε).

PROOF. The proof is given in BERGSTRA-KLOP [1]: Theorem 3.1 (iii) $\Rightarrow$ (i) followed by an application of the Countable Specification Lemma 5.1. (Note that the domain of ϕ contains only minimal algebras.) □

3. THE SPECIFICATION THEOREM

In this section we will state our theorem and give an informal outline of the formal proof which occupies Section 4.

3.1. THEOREM. *Let* ϕ: $\mathrm{Alg}(\Sigma) \longrightarrow \mathrm{Alg}(\Delta)$ *be a persistent parametrized data type with* $\mathrm{Dom}(\phi) = \mathrm{Alg}(\Sigma,E) \cap \mathrm{Sca}(\Sigma)$ *for some finite* E. *Then the following are equivalent:*

(i) ϕ *is effectively given,*

(ii) ϕ *has an algebraic specification* (Δ,H) *where* H *is r.e..*

First we will deal with the easy half (ii) $\Rightarrow$ (i) of the theorem.

PROOF of (ii) $\Rightarrow$ (i). Let $(\Sigma',E') \xrightarrow{\Sigma} A$ be a finite specification of a parameter algebra (with $\Sigma' \cap \Delta = \Sigma$). Then $(\Sigma' \cup \Delta, E' \cup H) \xrightarrow{\Delta} B$ with $(A,\gamma,B) \in \phi$. Because B has an r.e. specification, it is semi-computable. Using results from BERGSTRA-TUCKER [2] one uniformly computes from a specification $(\Sigma' \cup \Delta, E')$ and an r.e.-index of H a finite specification $(\Sigma^*,E^*) \xrightarrow{\Delta} B$ (which extends (Σ',E')). □

3.1.1. As to the proof of (i) $\Rightarrow$ (ii), we start with the following observation whose routine proof is omitted. First the

NOTATION. If $A \in \mathrm{Alg}(\Sigma)$, then $\langle A \rangle$ denotes $\langle A,\Sigma,\emptyset \rangle$, the subalgebra generated by the Σ-operations and constants. Note that $\langle A \rangle$ is a minimal algebra.

3.1.1.1. PROPOSITION. *Let* $A \in \mathrm{Alg}(\Sigma)$ *and let* e *be a closed conditional equation. Then* $A \models e \Longleftrightarrow \langle A \rangle \models e$.

Hence we can reduce satisfaction of an arbitrary conditional equation $e(\vec{x})$ in a Σ-algebra A, to satisfaction of closed conditional equations in some minimal subalgebras of A, as follows:

$$A \models e(\vec{x}) \Longleftrightarrow$$

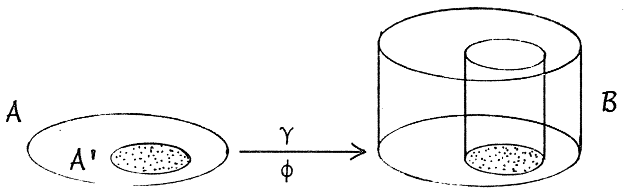
$$\forall \vec{a} \in A \quad A_{\vec{a}} \models e(\vec{c}) \Longleftrightarrow$$

$$\forall \vec{a} \in A \quad \langle A_{\vec{a}} \rangle \models e(\vec{c}).$$

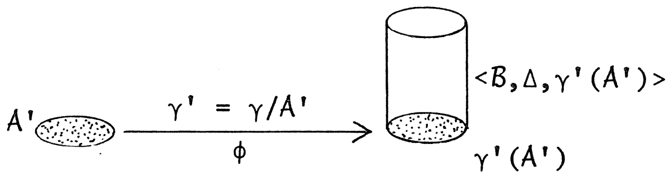
Here $A_{\vec{a}}$ is an expansion of A with constants $\vec{a}$ corresponding to $\vec{x}$, and $\vec{c}$ are constant symbols for $\vec{a}$.

3.1.2. Secondly, we observe (in Lemma 4.1) that a parametrized data type ϕ: Alg(Σ) $\longrightarrow$ Alg(Δ) with Dom(ϕ) = Alg(Σ,E) $\cap$ Sca(Σ) for some finite E, behaves well w.r.t. substructures of a parameter algebra A, as suggested by the following figure:

<u>if</u>



<u>then</u>

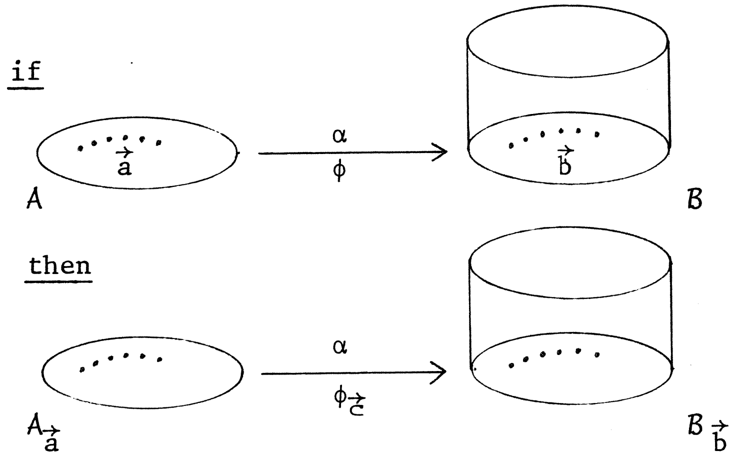


Here it should be remarked that we must restrict attention to those $A' \subseteq A$ which are still in Dom(ϕ). To ensure that, we need only require $A' \in$ Sca(Σ); for, $A' \in$ Alg(Σ,E) is trivially satisfied: conditional equations stay valid in a subalgebra. Note that if moreover A is *finitely generated*, i.e. $A' = \langle A,\Sigma,\vec{a}\rangle$ for a finite string $\vec{a}$ of elements in A, then:

$$A \in \mathrm{Sca}(\Sigma) \Rightarrow A' \in \mathrm{Sca}(\Sigma).$$

Indeed, this will be the case we will encounter.

3.1.3. Thirdly, from a parametrized data type ϕ: Alg(Σ) $\longrightarrow$ Alg(Δ) and a given string $\vec{c}$ of new constant symbols for the signature Σ, we define in the obvious way (see next figure) a parametrized data type $\phi_{\vec{c}}$: Alg($\Sigma_{\vec{c}}$) $\longrightarrow$ Alg($\Delta_{\vec{c}}$), where $\Sigma_{\vec{c}}$, $\Delta_{\vec{c}}$ is Σ,Δ plus the new constant symbols $\vec{c}$.



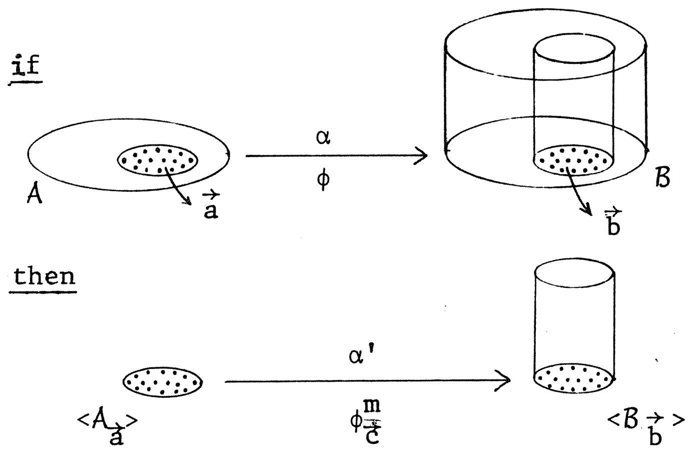
(Here the $\vec{a}$ are the interpretations of $\vec{c}$ in A, and $\vec{b}$ in B. Furthermore $\alpha(\vec{a}) = \vec{b}$.)

Not surprisingly, if ϕ is effectively given by (σ,ε), then the same holds for $\phi_{\vec{c}}$. (This is proved in 4.3.)

Now $\mathrm{Dom}(\phi_{\vec{c}}) = \mathrm{Alg}(\Sigma_{\vec{c}}, E) \cap \mathrm{Sca}(\Sigma_{\vec{c}})$. However, we will be only interested in the restriction of $\phi_{\vec{c}}$ to the class of minimal algebras of $\mathrm{Alg}(\Sigma_{\vec{c}}, E)$, i.e., the algebras $\langle A_{\vec{a}} \rangle$ from 3.1.1. Let $\phi^m_{\vec{c}}$ be this restriction. We already noted in 3.1.2:

$$A \in \mathrm{Dom}(\phi) \Rightarrow A_{\vec{a}} \in \mathrm{Dom}(\phi_{\vec{c}}) \Rightarrow \langle A_{\vec{a}} \rangle \in \mathrm{Dom}(\phi^m_{\vec{c}}) = \mathrm{ALG}(\Sigma_{\vec{c}}, E) \cap \mathrm{Sca}(\Sigma_{\vec{c}}) .$$

So $\phi^m_{\vec{c}}$ is as in the following figure:



3.1.4. In order to deal with all conditional equations $e(\vec{x})$ (where $\vec{x}$ might be arbitrarily long), we will use a countable set C of fresh constant symbols for Σ from which the $\vec{c}$ are taken.

It may be clear at this stage that the family of all $\phi^m_{\vec{c}}(\vec{c} \subseteq C)$ determines the original ϕ. (Section 4 proves this rigorously.) Moreover, $\phi^m_{\vec{c}}$ satisfies precisely the assumptions of Lemma 2.3; it is also effectively given by (σ,ε) and the domain has the required form. The persistency is obvious.

Hence $\phi^m_{\vec{c}}$ *has an algebraic specification* $(\Delta_{\vec{c}},H_{\vec{c}})$ *where* $H_{\vec{c}}$ *is an r.e. set of closed conditional equations.*

Now we remember that the $\vec{c}$ play in fact the role of variables (see 3.1.1.); so replacing the $\vec{c}$ again by corresponding variables x, we get $(\Delta,H_{\vec{x}/\vec{c}})$. As one may expect, taking together all these pieces of specifications to

$$(\Delta, \bigcup_{\vec{c} \subseteq C} H_{\vec{x}/\vec{c}}) = (\Delta,H)$$

yields the desired specification of ϕ. The proof that (Δ,H) specifies ϕ correctly, requires some more work however:

3.1.5. Consider the diagram

$$\begin{tikzcd}
I(Alg(\Sigma',E')) = \mathcal{A}' \arrow[r] \arrow[d, "\Sigma"] & \mathcal{B}' = I(Alg(\Sigma' \cup \Delta, E' \cup H)) \arrow[d, "?\ \Delta"] \\
\mathcal{A} \arrow[r, "\gamma", "\phi"'] & \mathcal{B}
\end{tikzcd}$$

where $(\mathcal{A},\gamma,\mathcal{B}) \in \phi$. We have to prove that $\mathcal{B}'|_{\Delta} \cong \mathcal{B}$. Now without loss of generality, we may take $\mathcal{A}'$ and $\mathcal{B}$ such that we can appeal to the 'Joint Expansion Property' in Section 1.1 and the joint expansion $\mathcal{A}' \sqcup \mathcal{B}$ can be formed. So, trivially, $(\mathcal{A}' \sqcup \mathcal{B})|_{\Delta} = \mathcal{B}$, and we must only prove that

$$\mathcal{A}' \sqcup \mathcal{B} \cong \mathcal{B}' = I(Alg(\Sigma' \cup \Delta, E' \cup H)).$$

In other words, we must prove the correctness of the specification

$$(\Sigma' \cup \Delta, E' \cup H) \xrightarrow{\Sigma' \cup \Delta} \mathcal{A}' \sqcup \mathcal{B}.$$

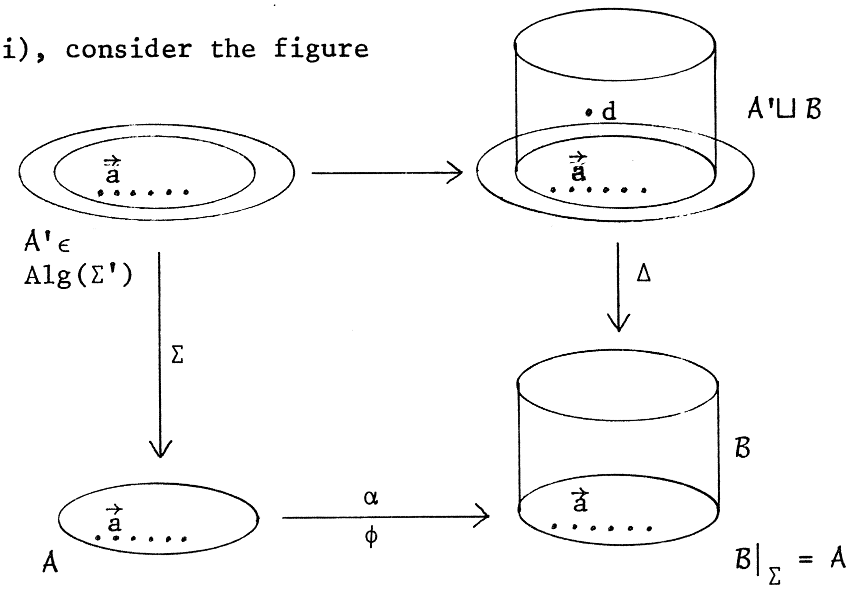
This amounts to proving

(i) *soundness*: $\mathcal{A}' \sqcup \mathcal{B} \models E' \cup H$

(ii) *completeness*: $\mathcal{A}' \sqcup \mathcal{B} \models s = t \Rightarrow E' \cup H \models s = t$, for all $s, t \in Ter^c(\Sigma' \cup \Delta)$.

We prove (i) in Section 4.3; it follows straightforwardly from the definition of H.

To prove (ii), consider the figure



Since $\mathcal{A}'$ is minimal, and $\mathcal{B}$ is generated from $\mathcal{B}|_\Sigma = \alpha(\mathcal{A}) = \mathcal{A}$, also $\mathcal{A}' \sqcup \mathcal{B}$ is minimal. I.e. every element in $\mathcal{A}' \sqcup \mathcal{B}$ is the denotation of a $\Sigma' \cup \Delta$-term. Something more can be said: since the $\vec{a}$ are denotated by Σ'-terms $\vec{s}$, the element d(generated from $\vec{a}$ by Δ-operations and constants) is denotated by a "$\Delta(\Sigma')$-term" $t(\vec{s})$, that is a Δ-term $t(\vec{x})$ in which the Σ'-terms $\vec{s}$ are substituted for $\vec{x}$.

Now if we can prove

(1) the completeness for the restricted class of $\Delta(\Sigma')$-terms and moreover,

(2) that each $\Sigma' \cup \Delta$-term is provably (from $E' \cup H$) equal to a $\Delta(\Sigma')$-term,

we are through. The proof of (1) is in Section 4.5, and of (2) in 4.7.

4. PROOF OF THE SPECIFICATION THEOREM

In this section we will give the formal details of the proof of Theorem 3.1 (ii) ⇒ (i) which we have already outlined in Section 3.

Let ϕ be an effective parametrized data type with $\mathrm{Dom}(\phi) =$

$Alg(\Sigma,E) \cap Sca(\Sigma)$ for some finite E, effectively given by (σ,ε).

We start with a lemma that explains the effect of ϕ on structures embedded in one another.

4.1. LEMMA. *Let* $(A,\gamma,B) \in \phi$, $A' \in Sca(\Sigma)$ *and* $A' \subseteq A$. *Then* $(A',\gamma',B') \in \phi$ *with* $\gamma' = \gamma \upharpoonright A'$ *and* $B' = \langle B,\Delta,\gamma'(A')\rangle$.

PROOF. Because $A' \subseteq A$ and $A' \models E$, together with $A' \in Sca(\Sigma)$ one finds $A' \in Dom(\phi)$. So there exist γ^*,B^* with $(A',\gamma^*,B^*) \in \phi$.

By (ii) of the definition of parametrized data type (Section 2) and the existence of an injective i embedding A' in A one derives the existence of λ such that the following diagram commutes:

$$\begin{tikzcd} A' \arrow[r, "\gamma^*"] \arrow[d, "i"'] \arrow[dr, "\gamma'"] & B \arrow[d, "\lambda"] \\ A \arrow[r, "\gamma'"] & B \end{tikzcd}$$

with $\gamma' = \gamma \circ i$ and λ injective.

Observe that $B^* = \langle B^*,\Delta,\gamma^*(A')\rangle$ by definition of parametrized data type, and that $\lambda(B^*) = \langle B,\Delta,\lambda\gamma^*(A')\rangle = \langle B,\Delta,\gamma \circ i(A')\rangle = B'$. It follows that the diagram

$$\begin{tikzcd} A' \arrow[r, "\gamma^*"] \arrow[d, "i"'] & B^* \arrow[d] \\ A' \arrow[r, "\gamma'"] & B' \end{tikzcd}$$

displays a congruence, whence $(A',\gamma',B') \in \phi$. □

4.2. Let C be a set of new constants for sorts of Σ, not occurring in Δ, in such a way that for each sort countably many new constants are introduced.

Furthermore, let $\Sigma_{\vec{c}}$, $\Delta_{\vec{c}}$ denote the result of augmenting Σ,Δ with a finite subset $\vec{c}$ of C. For finite $\vec{c} \subseteq C$ we define a parametrized data type $\phi_{\vec{c}}$ with domain $Alg(\Sigma_{\vec{c}},E) \cap Sca(\Sigma_{\vec{c}})$ and range in $Alg(\Delta_{\vec{c}})$ as follows:

$$(\mathcal{A},\gamma,\mathcal{B}) \in \phi_{\vec{c}} \text{ iff (i) } \mathcal{A} \in \mathrm{Alg}(\Sigma_{\vec{c}},E) \cap \mathrm{Sca}(\Sigma_{\vec{c}}),\ \mathcal{B} \in \mathrm{Alg}(\Delta_{\vec{c}})$$

$$\text{(ii)} (\mathcal{A}|_{\Sigma},\gamma,\mathcal{B}|_{\Delta}) \in \phi .$$

4.3. Restricting $\phi_{\vec{c}}$ to $\mathrm{ALG}(\Sigma_{\vec{c}},E) \cap \mathrm{Sca}(\Sigma_{\vec{c}})$ we obtain a parametrized data type $\phi^m_{\vec{c}}$ with range in $\mathrm{ALG}(\Delta_{\vec{c}})$. (Here the target algebras are indeed minimal, because they are generated from minimal parameter algebras.) Now $\phi^m_{\vec{c}}$ turns out to be effectively given by (σ,ε), just as ϕ itself is. This is evident from the following diagram:

$$\begin{tikzcd}
(\Sigma',E') \arrow[r, "{(\sigma,\varepsilon)}"] \arrow[d, "\Sigma_{\vec{c}}"] & (\Sigma'',E'') \arrow[d, "\Delta_{\vec{c}}"] \\
\mathcal{A} \arrow[r, "\phi_{\vec{c}}"] \arrow[d, "\Sigma"] & \mathcal{B} \arrow[d, "\Delta"] \\
\mathcal{A}' \arrow[r, "\phi"] & \mathcal{B}'
\end{tikzcd}$$

(Here it is essential that (σ,ε) is monotonic, from which it follows that $\Sigma'' \supseteq \Sigma_{\vec{c}} \cup \Delta = \Delta_{\vec{c}}$ because $\Sigma' \supseteq \Sigma_{\vec{c}}$.)

4.4. Applying Lemma 2.3 we obtain a specification $(\Delta_{\vec{c}},H_{\vec{c}})$ for $\phi^m_{\vec{c}}$ with $H_{\vec{c}}$ consisting of an r.e. set of closed conditional equations. $H_{\vec{c}}$ is uniformly r.e. in $(\vec{c},\sigma,\varepsilon)$.

Let x_c be a new variable for each $c \in C$ of the same sort. Write $H_{\vec{c}} = \{e^i_{\vec{c}} \mid i \in \omega\}$, and let $e^i_{\vec{x}/\vec{c}}$ be the result of substituting x_c for each occurrence of a constant symbol c (from C) in $e^i_{\vec{c}}$.
Obtain $H_{\vec{x}/\vec{c}} = \{e^i_{\vec{x}/\vec{c}} \mid i \in \omega\} = \{e^i_{\vec{x}/\vec{c}} \mid e^i_{\vec{c}} \in H_{\vec{c}}\}$. Note that $H_{\vec{x}/\vec{c}}$ is a set of conditional equations over the signature Δ. Taking the union of all specifications thus obtained one finds (Δ,H) with

$$H = \bigcup_{\vec{c} \subseteq C} H_{\vec{x}/\vec{c}} .$$

From the uniformity of finding $H_{\vec{c}}$ from $\vec{c}$ it follows that H is r.e.. Thus (Δ,H) is a specification of the required format.

4.5. CLAIM. (Δ,H) specifies ϕ. To show this, let (Σ',E') be a finite specification for $A \in \mathrm{Dom}(\phi)$, with $\Sigma' \cap \Delta = \Sigma$. Choose $(A,\gamma,B) \in \phi$. We must establish that the triples (A,γ,B) and $(I(\mathrm{Alg}(\Sigma',E'))|_{\Sigma}, \imath, I(\mathrm{Alg}(\Sigma' \cup \Delta, E' \cup H))|_{\Delta})$ are congruent.

We may assume that A is identical to $A'|_{\Sigma}$ with $A' = I(\mathrm{Alg}(\Sigma',E')$ and that $B|_{\Sigma} = A$ (whence $\gamma = \mathrm{id}$) and further that the domains corresponding to sorts of A' and B not named in Σ are pairwise disjoint.

Let $A' \sqcup B$ be the joint expansion of A' and B. Note that $A' \sqcup B$ is a minimal $\Sigma' \cup \Delta$-algebra. To prove

$$(\Sigma' \cup \Delta, E' \cup H) \xrightarrow{\Sigma' \cup \Delta} A' \sqcup B$$

it suffices to derive soundness and completeness of $E' \cup H$.

(i) Soundness. Let $e \in E' \cup H$. If $e \in E'$ then $A' \models e$ and so $A' \sqcup B \models e$. If $e \in H$, choose $\vec{c} \subseteq C$ such that $e = e_{\vec{x}/\vec{c}} \in H_{\vec{x}/\vec{c}}$. Take a set of values $\vec{a}$ in $A' \sqcup B$ of suitable sorts corresponding to $\vec{c}$. Note that $\vec{a}$ must be from sorts (Σ); hence $\vec{a} \subseteq A \subseteq A' \sqcup B$. We will show that $A' \sqcup B$ satisfies e in $\vec{a}$, i.e.

$$(A' \sqcup B)_{\vec{a}} \models e(\vec{c}).$$

Now consider $\langle A_{\vec{a}} \rangle$ and $\langle B_{\vec{a}} \rangle$. From Lemma 4.1 and the definition of $\phi^m_{\vec{c}}$ we find that

$$(\langle A_{\vec{a}} \rangle, \mathrm{id}, \langle B_{\vec{a}} \rangle) \in \phi^m_{\vec{c}} .$$

Because $H_{\vec{c}}$ specifies $\phi^m_{\vec{c}}$, we have $\langle B_{\vec{a}} \rangle \models H_{\vec{c}}$. Especially $\langle B_{\vec{a}} \rangle \models e(\vec{c})$; and since $\langle B_{\vec{a}} \rangle \subseteq (A' \sqcup B)_{\vec{a}}$, also $(A' \sqcup B)_{\vec{a}} \models e(\vec{c})$. □

(ii) Completeness for $\Delta(\Sigma')$-terms. Let $A' \sqcup B \models t = r$ where $t = r$ is a closed equation. If $t,r \in \mathrm{Ter}^c(\Sigma')$, there is no problem: since E' specifies A', we have $E' \vdash t = r$. Otherwise, we restrict our attention to closed equations

$t = r$ of the form $t = t(\tau_1 \ldots, \tau_k)$, $r = r(\tau_1, \ldots, \tau_k)$ where $t(x_1, \ldots, x_k)$, $r(x_1, \ldots, x_k) \in \mathrm{Ter}(\Delta)$ and $\tau_i \in \mathrm{Ter}(\Sigma')$, $i = 1, \ldots, k$. Here it is not required that all x_i $(i=1,\ldots,k)$ do occur in $t(\vec{x})$ and $r(\vec{x})$.
(Such t,r are called $\Delta(\Sigma')$-terms; see Section 4.7) Moreover, we require the $\vec{x}$ to be variables for Σ-sorts.

So suppose $A' \sqcup B \models t = r$; we will prove that $E' \cup H \models t = r$. Let $\vec{a} = (a_1, \ldots, a_k)$ be the values of $(\tau_1, \ldots, \tau_k)$ in A; they are also the values of $(\tau_1, \ldots, \tau_k)$ in B and in $A' \sqcup B$. As before, $A_{\vec{a}}$ and $B_{\vec{a}}$ are the expansions of A, B by adding $\vec{a}$ as constants. The corresponding signatures are $\Sigma_{\vec{c}}$ resp. $\Delta_{\vec{c}}$. Further, $\langle A_{\vec{a}} \rangle$ and $\langle B_{\vec{a}} \rangle$ are again the minimal substructures. From $A' \sqcup B \models t = r$ we have $B \models t = r$, hence $B_{\vec{a}} \models t(\vec{c}) = r(\vec{c})$ and $\langle B_{\vec{a}} \rangle \models t(\vec{c}) = r(\vec{c})$ (Prop. 3.1.1.1.)

Let $\equiv_{\vec{c}}$ abbreviate $\equiv_{\langle A_{\vec{a}} \rangle}$. Clearly $(\Sigma_{\vec{c}}, \equiv_{\vec{c}})$ specifies $\langle A_{\vec{a}} \rangle$; and because $(\Delta_{\vec{c}}, H_{\vec{c}})$ specifies $\phi^m_{\vec{c}}$ we have the following diagram:

$$\begin{tikzcd}
(\Sigma_{\vec{c}}, \equiv_{\vec{c}}) \arrow[r] \arrow[d, "\Sigma_{\vec{c}}"] & (\Delta_{\vec{c}}, \equiv_{\vec{c}} \cup H_{\vec{c}}) \arrow[d, "\Delta_{\vec{c}}"] \\
\langle A_{\vec{a}} \rangle \arrow[r, "\phi^m_{\vec{c}}"'] & \langle B_{\vec{a}} \rangle
\end{tikzcd}$$

From $\langle B_{\vec{a}} \rangle \models t(\vec{c}) = r(\vec{c})$ it follows that $\equiv_{\vec{c}} \cup H_{\vec{c}} \vdash t(\vec{c}) = r(\vec{c})$. A fortiori: $\equiv_{\vec{c}} \cup H_{\vec{x}/\vec{c}} \vdash t(\vec{c}) = r(\vec{c})$. Now let $\equiv_{\vec{\tau}/\vec{c}}$ be the result of substituting τ_i for $c_i (i=1,\ldots,k)$ in the equations in $\equiv_{\vec{c}}$. Then also

$$\equiv_{\vec{\tau}/\vec{c}} \cup H_{\vec{x}/\vec{c}} \vdash t(\vec{\tau}) = r(\vec{\tau}).$$

Now the equations in $\equiv_{\vec{\tau}/\vec{c}}$ are closed Σ'-equations, true in A'; hence they are derivable from E', the specification of A'. So we have

$$E' \cup H_{\vec{x}/\vec{c}} \vdash t(\vec{\tau}) = r(\vec{\tau}). \qquad \square$$

4.6. Intermezzo: $\Sigma_1(\Sigma_2)$-terms.

Let Σ_1,Σ_2 be extension signatures of Σ_0 such that $\Sigma_1 \cap \Sigma_2 = \Sigma_0$. We will define $\mathrm{Ter}(\Sigma_1(\Sigma_2))$, a subset of $\mathrm{Ter}(\Sigma_1 \cup \Sigma_2)$; and for $t \in \mathrm{Ter}(\Sigma_1 \cup \Sigma_2)$ we will define the $\Sigma_1|\Sigma_2$-*degree* of t. In a $\Sigma_1 \cup \Sigma_2$-term t the symbols (i.e. the names of functions and constants) from $\Sigma_0,\Sigma_1,\Sigma_2$ can occur in a complex 'mixed' fashion, see Example 4.6.4; the $\Sigma_1|\Sigma_2$-degree is a measure of this complexity.

Let $t \in \mathrm{Ter}(\Sigma_1 \cup \Sigma_2)$ and let Tree (t) be its formation tree, written such that the head operator of t is the top label of the tree. We will refer to the symbols from Σ_0 as 0-synbols, from $\Sigma_1-\Sigma_0$ as I-symbols and from $\Sigma_2-\Sigma_0$ as II-symbols. Here 0,I,II are called labels of symbols. Now to each branch α in Tree (t) we associate the tuple of labels of the symbols occurring in α, 'reading' α starting at the top of Tree (t). (See Example 4.6.4.) From each such tuple, e.g. (I,0,0,II,I,0,I,II,0), we compute the number of alternations from a I-to a II-label and vice versa, disregarding the 0-labels. In the example just given, this *alternation number* is 3.

4.6.1. DEFINITION. The $\Sigma_1|\Sigma_2$-degree of t is the multiset of alternation numbers of all branches in Tree (t). The degrees are ordered by the usual multiset ordering.

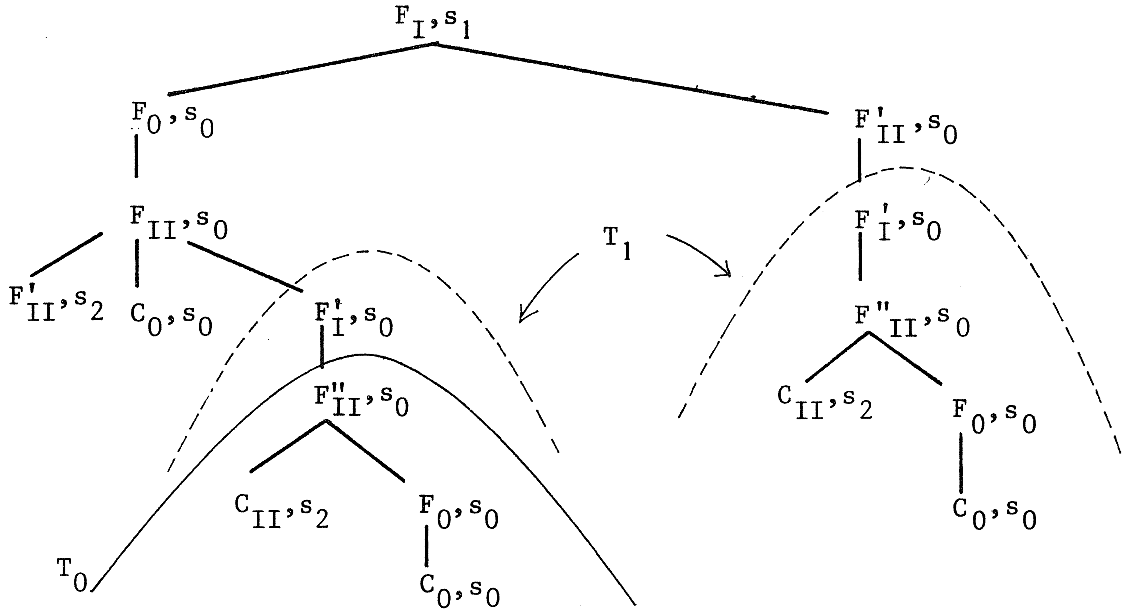
4.6.2. DEFINITION. $\mathrm{Ter}(\Sigma_1(\Sigma_2))$, the set of $\Sigma_1(\Sigma_2)$-terms, is the union of $\mathrm{Ter}(\Sigma_2)$ and the set of results $t(\vec{s})$ of substitutions of Σ_2-terms $\vec{s}$ into Σ_1-terms $t(\vec{x})$.

4.6.3. REMARK. (i) $\mathrm{Ter}(\Sigma_1 \cup \Sigma_2) \supseteq \mathrm{Ter}(\Sigma_1\Sigma_2)) \supseteq \mathrm{Ter}(\Sigma_1) \cup \mathrm{Ter}(\Sigma_2)$.
(ii) t is a $\Sigma_1(\Sigma_2)$-term iff in Tree(t) no I-symbol occurs below a II-symbol. (So along each branch there is at most one alternation allowed, viz. from a I- to a II-symbol, disregarding 0-symbols.)

4.6.4. EXAMPLE. Σ_0 has *sorts* s_0, *functions* $F_0: s_0 \longrightarrow s_0$, *constants* $C_0 \in s_0$;
$\Sigma_1-\Sigma_0$ has *sorts* s_1, *functions* $F_I: s_0 \times s_0 \longrightarrow s_1$;
$\Sigma_2-\Sigma_0$ has *sorts* s_2, *functions* $F_{II}: s_2 \times s_0 \times s_0 \longrightarrow s_0$, $F'_{II}: s_0 \longrightarrow s_0$,
$F''_{II}: s_2 \times s_0 \longrightarrow s_0$, *constants* $C_{II} \in s_2$.

Let $t \in \mathrm{Ter}(\Sigma_1 \cup \Sigma_2)$ have the following tree (where next to each function and constant symbol also its target sort is indicated): (see figure next page).

Here the tuple corresponding to e.g. the rightmost branch is (I,II,I,II,0,0), with alternation number 3. Now the $\Sigma_1|\Sigma_2$-degree of t is {1,1,3,3,3,3}.



4.6.4.1. <u>REMARK</u>. Note that if a subterm having the tree T_0(as indicated in Example 4.6.4), denoting an s_0-element, is replaced by a Σ_0-term denoting the same element (if such a term exists), then this elimination of the 'foreign' II-symbols F''_{II}, C_{II} results in a decreased $\Sigma_1|\Sigma_2$-degree , viz. {1,1,2,2,3,3}. Furthermore, if the twice occurring subtree T_1 is replaced by a Σ_0-term, the result would be a $\Sigma_1(\Sigma_2)$-term.

It is important to note the following obvious fact:

4.6.5. <u>PROPOSITION</u>. *If in a branch* α *of Tree* (t), $t \in \mathrm{Ter}(\Sigma_1 \cup \Sigma_2)$, *a* II-*symbol* F_{II} *is followed immediately by a* I-*symbol* G_I*(disregarding* 0-*symbols), i.e. the tuple of* α *is*

$$(\text{---},II,0,0,\ldots,0,I,\text{---}) \quad (k \geq 0 \text{ times } 0)$$

where the displayed II,I *are the labels of* F_{II},G_I, *then the target sort of* G_I *must be a* Σ_0-*sort.* □

4.7. It remains to be shown that each $\Sigma' \cup \Delta$-term is provably (from $E' \cup H$) equal to some $\Delta(\Sigma')$-term.

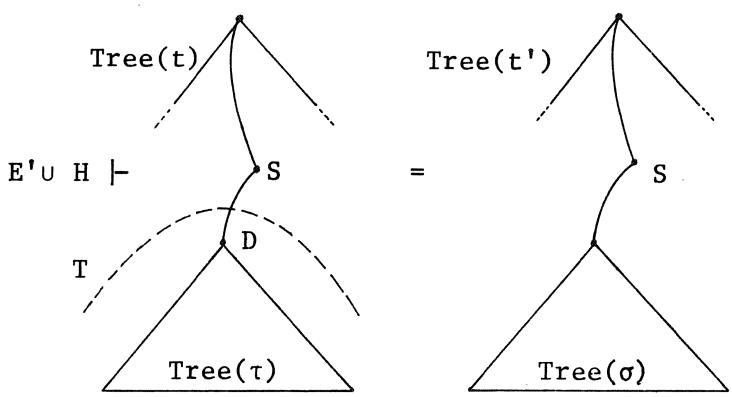
Let $t \in Ter(\Sigma' \cup \Delta)$. Consider Tree (t). If $t \notin Ter(\Delta(\Sigma'))$, then there is a $(\Delta-\Sigma)$-function or constant symbol, say D, occurring below an $(\Sigma'-\Sigma)$-function or constant symbol, say S.

Now we can find in Tree (t) a pair S,D such that

(i) D is below S,

(ii) S is immediately followed by D (disregarding Σ-symbols),

(iii) the pair S,D is a lowest pair with these properties.



Then, as we observed in Proposition 4.6.5 the target sort of D must be a Σ-sort. Let T be the subtree headed by D and let τ be the corresponding term. Since τ denotes an element of a Σ-sort, $\mathcal{A}' \sqcup \mathcal{B} \models \tau = \sigma$ for some $\sigma \in Ter(\Sigma')$. Noting that $\sigma, \tau \in Ter(\Delta(\Sigma'))$, we have by the completeness of $E' \cup H$ for $\Delta(\Sigma')$-terms, as proved in 4.5:

$$E' \cup H \vdash \tau = \sigma.$$

Now let t' be t where τ is replaced by σ. Then also

$$E' \cup H \vdash t = t',$$

and the $\Delta|\Sigma'$-degree of t' is less than that of t. Continuing this procedure we find

$$E' \cup H \vdash t = t' = t'' = \ldots = s$$

for some $\Delta(\Sigma')$-term s. □

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